TA NOTES

Find the general solution of the given differential equation

(1). $y'' + 2y' - 3y = 0$

(3). $6y'' - y' - y = 0$

(5). $y'' + 5y' = 0$

(6). $4y^{''} - 9y = 0$

(7). $y'' - 9y' + 9 = 0$

(8). $y^{''} - 2y^{'} - 2y = 0$

Answer: (1). The characteristic equation is

$$r^2 + 2r - 3 = (r-1)(r+3) = 0$$

Thus the possible values of r are $r_1 = 1$ and $r_2 = -3$, and the general solution of the equation is

$$y(t) = c_1 e^t + c_2 e^{-3t}.$$

(3). The characteristic equation is

$$6r^2 - r - 1 = (2r-1)(3r+1) = 0$$

Thus the possible values of r are $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{3}$, and the general solution of the equation is

$$y(t) = c_1 e^{-\frac{t}{3}} + c_2 e^{\frac{t}{2}}.$$

(5). The characteristic equation is

$$r^2 + 5r = r(r+5) = 0$$

Thus the possible values of r are $r_1 = 0$ and $r_2 = -5$, and the general solution of the equation is

$$y(t) = c_1 + c_2 e^{-5t}.$$

(6). The characteristic equation is

$$4r^2 - 9 = 0$$

Thus the possible values of r are $r = \frac{3}{2}$ and $r = -\frac{3}{2}$, and the general solution of the equation is

$$y(t) = c_1 e^{\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t}.$$

(7). The characteristic equation is

$$r^2 - 9r + 9 = (r - \frac{9+3\sqrt{5}}{2})(r - \frac{9-3\sqrt{5}}{2}) = 0$$

Thus the possible values of r are $r_1 = \frac{9+3\sqrt{5}}{2}$ and $r_2 = \frac{9-3\sqrt{5}}{2}$, and the general solution of the equation is

$$y(t) = c_1 e^{\frac{9+3\sqrt{5}}{2}t} + c_2 e^{\frac{9-3\sqrt{5}}{2}t}.$$

(8). The characteristic equation is

$$r^2 - 2r - 2 = (r - 1 - \sqrt{3})(r - 1 + \sqrt{3}) = 0$$

Thus the possible values of r are $r_1 = 1 - \sqrt{3}$ and $r_2 = 1 + \sqrt{3}$, and the general solution of the equation is

$$y(t) = c_1 e^{(1-\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t}.$$

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Find the general solution of the given differential equation. Sketch the graph of the solution and describe its behavior as t increases

(10). $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$

(13). $y^{''} + 5y^{'} + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$

(16). $4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1$

Answer: (10). The characteristic equation is

$$r^2 + 4r + 3 = 0$$

Thus the possible values of r are $r = -1$ and $r = -3$, and the general solution of the equation is

$$y(t) = c_1 e^{-t} + c_2 e^{-3t}.$$

From the initial value, we have

$$\begin{aligned} c_1 + c_2 &= 2 \\ -c_1 - 3c_2 &= -1, \end{aligned} \tag{1}$$

hence $c_1 = \frac{5}{2}$ and $c_2 = -\frac{1}{2}$. The solution of the equation is

$$y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}.$$

$y(t) \longrightarrow 0$, as $t \longrightarrow \infty$.

(13). The characteristic equation is

$$r^2 + 5r + 3 = 0$$

Thus the possible values of r are $r = \frac{-5+\sqrt{13}}{2}$ and $r = \frac{-5-\sqrt{13}}{2}$, and the general solution of the equation is

$$y(t) = c_1 e^{\frac{-5+\sqrt{13}}{2}t} + c_2 e^{\frac{-5-\sqrt{13}}{2}t}.$$

From the initial value, we have

$$\begin{aligned} &c_1 + c_2 = 1 \\ &\frac{-5+\sqrt{13}}{2}c_1 + \frac{-5-\sqrt{13}}{2}c_2 = 0, \end{aligned} \tag{2}$$

hence $c_1 = \frac{13+5\sqrt{13}}{26}$ and $c_2 = \frac{13-5\sqrt{13}}{26}$. The solution of the equation is

$$y(t) = \frac{13+5\sqrt{13}}{26}e^{\frac{-5+\sqrt{13}}{2}t} + \frac{13-5\sqrt{13}}{26}e^{\frac{-5-\sqrt{13}}{2}t}.$$

$y(t) \longrightarrow 0$, as $t \longrightarrow \infty$.

(16). The characteristic equation is

$$4r^2 - 1 = 0$$

Thus the possible values of r are $r = 1/2$ and $r = -1/2$, and the general solution of the equation is

$$y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t}.$$

From the initial value, we have

$$\begin{aligned} &c_1 e^{-1} + c_2 e = 1 \\ &\frac{1}{2}c_1 e^{-1} - \frac{1}{2}c_2 e = -1, \end{aligned} \tag{3}$$

hence $c_1 = -\frac{e}{2}$ and $c_2 = \frac{3e^{-1}}{2}$. The solution of the equation is

$$-\frac{e}{2}e^{\frac{1}{2}t} + \frac{3e^{-1}}{2}e^{-\frac{1}{2}t}.$$

$y(t) \longrightarrow -\infty$, as $t \longrightarrow \infty$.

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(18). Find a differential equation whose general solution is $y = c_1 e^{-t/2} + c_2 e^{-2t}$

Answer: Suppose the equation is

$$y''(t) + ay'(t) + by(t) = 0$$

then the characteristic equation is

$$r^2 + ar + b = 0.$$

Since $r_1 = -\frac{1}{2}$ and $r_2 = -2$ must be two root of the characteristic equation, we have $a = -(r_1 + r_2) = \frac{1}{2}$ and $b = r_1 r_2 = 1$. Then the equation

$$y''(t) + \frac{1}{2}y'(t) + y(t) = 0$$

is the one we want.

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2. Solve the following initial value problem and indicate the interval of existence of solution

$$\begin{cases} (2x-y)dx + (2y-x)dy = 0 \\ y(1) = 3 \end{cases}$$

Answer: Compute

$$\frac{\partial(2x-y)}{\partial y} = -1 = \frac{\partial(2y-x)}{\partial x}$$

So the equation is exact. It is easy to see $x^2 - xy + y^2 = C$. By $y(1) = 3$, we know that $C = 7$. So

$$y = \frac{x \pm \sqrt{28 - 3x^2}}{2}$$

Again by $y(1) = 3$,we know that the solution is

$$y = \frac{x + \sqrt{28 - 3x^2}}{2}$$

We obtain $-\sqrt{28/3} \leq x \leq \sqrt{28/3}$ by $28 - 3x^2 \geq 0$. However when $x = \pm\sqrt{28/3}$, we have $2y - x = 0$ and so $(2x-y)dx = 0$, it is impossible! So the interval of existence is $(-\sqrt{28/3}, \sqrt{28/3})$

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